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Rank-Envy-Freeness in Roommate Matchings

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Abstract. In the roommate problem, pairs of agents must be formed, based on ordinal preferences of the agents over each other. In this article, we examine fair roommate matchings by relaxing envy-freeness to account for justified envy based on the rank in the agents' preferences. A rank-envy-free matching prevents that an agent prefers the partner of another agent whereas she has ranked it better. Although this requirement is pretty weak in house allocation [9], we show that it is more demanding in the roommate setting. We study parameterizations of rank-envy-freeness, as well as further natural relaxations of this concept. We also investigate the connections between the family of rank-based fairness criteria and known optimality or stability concepts.

1 Introduction

Matching under preferences is a very attractive framework that captures a large number of real-world applications. This research area has gained much attention in computer science, in particular in algorithmic game theory and computational social choice [28, 30]. The setting where pairs of agents must be formed, without distinction among the agents, is known as the *roommate setting* [22]. This is the non-bipartite version of the marriage setting, where the agents are partitioned in two types and only agents of different types can be matched. The roommate setting can be seen as a specific *hedonic game* [7], where coalitions of agents of size two must be formed.

This matching problem has been initially studied under the prism of stability [25], in the line of the seminal work of Gale and Shapley on the *stable marriage problem* [22]. Given ordinal agents' preferences, the aim is to find a matching that is stable w.r.t. blocking pairs of agents (BP) who prefer to be matched together than with their current partner in the matching. Alternative notions of stability have been introduced since then, and in particular the one of *exchange-stability* (also known as *swap-stability*) [6, 14].

Another important direction of research in matching problems concerns the search for *optimal matchings*, aiming to ensure the global satisfaction of the agents. In the context of ordinal preferences, the most prominent optimality concept is Pareto-optimality [5]. Yet, since many Pareto-optimal matchings exist, refinements have been proposed, e.g., *rank-maximality* [26] which aims to lexicographically optimize the ranks of the matched agents, and *popularity* [18], which is based on pairwise comparisons of matchings by the agents.

Alternatively to stability or optimality, we adopt another view in this article by exploring *fairness* issues in roommate matchings. Contrary to optimality, fairness does not focus on the global satisfaction but on the individual feeling of fair treatment. While an unstable matching is typically also unfair, the two notions are independent because feeling of fair treatment does not imply that the solution is immune to local perturbations and agents may still feel frustrated in a stable solution [34]. Although fairness has been widely investigated in resource allocation [12, 32], this topic has been less explored in settings matching agents with other agents. The main development in the literature concerns *justified envy-freeness* in school choice [2], which aims to avoid that a student prefers to be matched with a given college that also prioritizes her. The analogous notion of fairness has also been studied in hedonic games [35]. Other works concern the marriage setting to counterbalance the inherent difference of treatment between the two types of agents in stable matchings [31]. In roommate matchings, we can particularly cite the rank-fairness notion [27] imposing that partners assign the same rank to each other.

In this article, we focus on another notion of fairness called *rank-envy-freeness (r-EF)*, which is a relaxation of envy-freeness [20] that requires that no agent prefers to be matched with the partner of another agent whereas she has ranked the desired agent at a better position in her preference ranking. This concept is particularly relevant in an ordinal context and is related to the notion of *justified envy*, but considers the justification for envy w.r.t. the preferences of the envious agent: Alice can legitimately feel discriminated if Bob got matched with Charlie, who is his second choice, whereas Alice has reported that Charlie was her first choice. This notion has been introduced in school choice mechanism design under the name *favoring higher ranks* [29], and has been mainly studied in random assignment (of agents to objects) [23, 33]. Rank-envy-freeness has also been recently investigated [9] in house allocation [1], a setting where each agent must be matched with a single object. This latter work proves that both rank-maximality and popularity imply r-EF, guaranteeing the existence of an r-EF allocation, while no implication relation holds with Pareto-optimality. It also generalizes the notion to *rank$_k$-envy-freeness (r$_k$-EF)* and proves that popularity is equivalent to r$_1$-EF when there are as many objects as agents.

In this article, we investigate rank-envy-freeness in the particular roommate setting where agents have strict ordinal preferences over all other agents and where every agent must be matched. Preference restrictions are sometimes considered to strengthen our results. We show that, contrary to house allocation, an r-EF matching does not always exist and the previous implications with optimality do not hold anymore. We also examine r$_k$-EF matchings and show that r$_1$-EF is no longer equivalent to popularity but keeps the characterization of popularity for house allocation [3], making it decidable in polynomial time, contrary to popularity [19, 24]. We further introduce new relaxations of r-EF, namely *weak rank-envy-freeness (wr-EF)*, *BP rank-envy-freeness (r^{BP}-EF)* and *weak BP rank-envy-freeness (wr^{BP}-EF)*, and show that we can always construct a wr^{BP}-EF matching in polynomial time. This finding contrasts with the NP-completeness of

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the decision problems related to the existence of matchings satisfying r-EF, wr-EF or r[BP]-EF. We determine the exact implication relations between all these *rank-based envy-freeness* criteria and known optimality or stability criteria; they are summarized in Figure 1. Finally, we empirically evaluate the existence of such fair matchings.

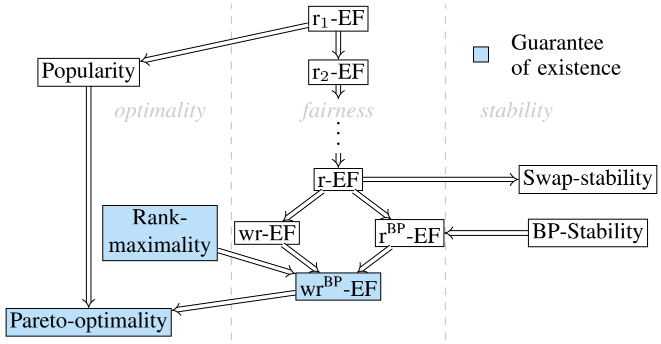


Figure 1: Implication relations between rank-based envy-freeness criteria and optimality or stability criteria.

2 Preliminaries

For an integer k, let $[k] := \{1, \ldots, k\}$. We are given a set of agents $N = [n]$ where n is even. Each agent $i \in N$ is associated with a linear order $\succ_i$ over $N \setminus \{i\}$ representing her strict ordinal preferences over the other agents. A preference profile $\succ$ is the set of all linear orders $\succ_i$ for all agents $i \in N$. A roommate instance is thus simply a pair $(N, \succ)$. A matching $\sigma : N \to N$ is a bijection over N.

2.1 Preference restrictions

We sometimes consider restrictions in the preference profile, that can capture a certain correlation in the agents' preferences.

A preference profile $\succ$ is *single-peaked* [11] if there exists a linear order $<$ over N such that for each agent $i \in N$ and each triple of agents $j, k, \ell \in N \setminus \{i\}$ with $j < k < \ell$ or $\ell < k < j$, we have $j \succ_i k$ implies $k \succ_i \ell$. A particular case is when agents consider their own position as an ideal position in the axis. A preference profile $\succ$ is *narcissistically single-peaked* [8] if $\succ$ is single-peaked w.r.t. axis $<$ and, for every agent, her most preferred agent is directly adjacent to her in axis $<$, either on her "left" or on her "right".

Instead of a common axis, a global evaluation of the matches can be considered. A preference profile $\succ$ is *globally-ranked* [4] if there exists a global order $\rhd$ over all possible agent pairs such that for every $i \in N$ and $j, k \in N \setminus \{i\}$, $j \succ_i k$ iff $\{i, j\} \rhd \{i, k\}$; i.e., each agent prefers those with whom she forms an objectively better pair.

Finally, we consider a stronger restriction which satisfies both narcissistic single-peakedness and global-rankedness. A preference profile $\succ$ is *1-Euclidean* [17] if there exists an embedding $E : N \to \mathbb{R}$ such that for every agent $i \in N$ and two agents $j, k \in N \setminus \{i\}$, $j \succ_i k$ iff $|E(i) - E(j)| < |E(i) - E(k)|$. The idea is that the agents prefer those that are closer to them on a given common scale.

2.2 Stable matchings

Stability is initially defined w.r.t. a *blocking pair*: agents i and j form a blocking pair in matching σ if they prefer to be matched together than with their partner in σ, i.e., $j \succ_i \sigma(i)$ and $i \succ_j \sigma(j)$. A matching σ is *blocking-pair (BP-)stable* if there is no blocking pair in σ.

Another notion of stability comes from the notion of swap. A matching σ is *swap-stable* if there is no pair of agents i and j who prefer to swap their partner, i.e., $\sigma(j) \succ_i \sigma(i)$ and $\sigma(i) \succ_j \sigma(j)$.

2.3 Optimal matchings

One of the most classical optimality notion is Pareto-optimality. A matching σ' Pareto-dominates a matching σ if $\sigma'(i) \succeq_i \sigma(i)$ for every agent i and there exists an agent j such that $\sigma'(j) \succ_j \sigma(j)$. A matching σ is *Pareto-optimal* if no matching σ' Pareto-dominates it.

The Pareto criterion can be strengthened by considering a lexicographic maximization of the agents' satisfaction given by the rank of their partner in the matching. The *signature* s_σ of matching σ is the $(n-1)$-vector giving the number of agents $s_\sigma(k)$ assigned to their k^{th} most preferred agent in σ, for all $k \in [n-1]$. Signature s_σ is lexicographically strictly greater than signature $s_{\sigma'}$, denoted by $s_\sigma >_{lex} s_{\sigma'}$, if there exists an index $i \in [n-1]$ such that $s_\sigma(i') = s_{\sigma'}(i')$ for all $i' < i$ and $s_\sigma(i) > s_{\sigma'}(i)$. A matching σ is *rank-maximal* if there is no matching σ' such that $s_{\sigma'} >_{lex} s_\sigma$.

Another direction to strengthen Pareto-optimality is by considering votes between matchings. Agent i prefers matching σ to matching σ' if $\sigma(i) \succ_i \sigma'(i)$. A matching σ is more popular than a matching σ' if the number of agents who prefer σ to σ' is strictly greater than the number of agents who prefer σ' to σ. A matching σ is *popular* if there is no matching σ' more popular than σ.

3 Rank-Based Envy-Free Matchings

We study *fair* matchings w.r.t. rank-based envy-freeness notions.

3.1 Rank-envy-freeness

The classical notion of envy-freeness [20] is too strong in the roommate setting, because it requires that every agent is matched with her most preferred agent. Therefore, we consider *rank-envy-freeness*, a relaxation of envy-freeness which aims to prevent justified envy based on the rank. An agent i rank-envies another agent j in matching σ if i prefers the partner of j to her own whereas i ranks the desired partner in a better position in her preference ranking than agent j. Let $r_i : N \to [n-1]$ be the function giving the rank of an agent in $\succ_i$, i.e., $r_i(j) = |\{\ell \in N : \ell \succeq_i j\}|$, for agent $j \in N \setminus \{i\}$.

Definition 1 (Rank-envy-freeness (r-EF))**.** *A matching σ is r-EF if $\sigma(i) \succ_i \sigma(j)$ or $r_j(\sigma(j)) \leq r_i(\sigma(j))$, for every $i, j \in N$.*

A generalization of r-EF can be defined by considering a parameterization of the concept w.r.t. rank $0 \leq k < n-1$. The associated justification for envy is that the desired agent has not been ranked good enough (among the top k best agents) by the envied agent.

Definition 2 (Rank$_k$-envy-freeness (r$_k$-EF))**.** *A matching σ is r$_k$-EF if $\sigma(i) \succ_i \sigma(j)$ or $r_j(\sigma(j)) \leq \min\{r_i(\sigma(j)), k\}$ for every $i, j \in N$.*

Note that r$_k$-EF implies r$_{k'}$-EF for every $k' > k$. Moreover, r$_{n-2}$-EF is equivalent to r-EF and r$_0$-EF is equivalent to envy-freeness.

The next basic first observation can be made for r-EF matchings.

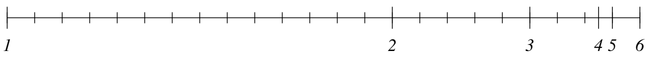
Observation 1. *Every agent that is ranked first by some agent must be matched in an r-EF matching with an agent who ranks her first.*

Contrary to house allocation where an r-EF allocation always exists [9], an r-EF matching may not exist in roommate instances, even under 1-Euclidean preferences, as shown in Example 1.

Example 1. *Let us consider a roommate instance with six agents who have the preferences below.*

$$\begin{array}{lccccccccc}
1: & \boxed{2} & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
2: & 3 & \succ & 4 & \succ & 5 & \succ & 6 & \succ & \boxed{1} \\
3: & 4 & \succ & 5 & \succ & \boxed{6} & \succ & 2 & \succ & 1 \\
4: & \boxed{5} & \succ & 6 & \succ & 3 & \succ & 2 & \succ & 1 \\
5: & \boxed{4} & \succ & 6 & \succ & 3 & \succ & 2 & \succ & 1 \\
6: & 5 & \succ & 4 & \succ & \boxed{3} & \succ & 2 & \succ & 1
\end{array}$$

The preferences are 1-Euclidean w.r.t. the following embedding:



By Observation 1, in an r-EF matching, agent 2 must be matched with agent 1 while agent 3 must be matched with agent 2, a contradiction. Hence, there cannot exist an r-EF matching in this instance.

3.2 Relaxations of rank-envy-freeness

We thus propose relaxations of r-EF, which strengthen the conditions for justified envy. First, if an agent is well-ranked by her current partner, she may feel flattered and would not dare to claim for a better partner, especially if she does not rank the desired agent at a better rank than what her current partner does for her. The associated notion of envy is as follows: agent i *strongly rank-envies* agent j in matching σ if $\sigma(j) \succ_i \sigma(i)$, $r_i(\sigma(j)) < r_j(\sigma(j))$, and $r_{\sigma(i)}(i) > r_i(\sigma(j))$.

Definition 3 (Weak rank-envy-freeness (wr-EF))**.** *A matching σ is wr-EF if $\sigma(i) \succ_i \sigma(j)$ or $r_j(\sigma(j)) \leq r_i(\sigma(j))$ or $r_{\sigma(i)}(i) \leq r_i(\sigma(j))$, for every $i, j \in N$.*

Example 1 (continued)**.** *The framed matching is wr-EF. Indeed, the only rank-envious agent is agent 2 (who is rank-envious towards agent 6), but she is matched with agent 1 who ranks her first, therefore agent 2 does not complain.*

Another direction for relaxing r-EF is to consider a justification of envy based on a blocking pair: the desired partner would also prefer to be matched with the envious agent. Agent i BP rank-envies agent j in matching σ if $\sigma(j) \succ_i \sigma(i)$, $r_i(\sigma(j)) < r_j(\sigma(j))$, and $i \succ_{\sigma(j)} j$.

Definition 4 (BP rank-envy-freeness (r^{BP}-EF))**.** *A matching σ is r^{BP}-EF if $\sigma(i) \succ_i \sigma(j)$ or $r_j(\sigma(j)) \leq r_i(\sigma(j))$ or $j \succ_{\sigma(j)} i$, for every $i, j \in N$.*

Example 1 (continued)**.** *The framed matching is r^{BP}-EF. Indeed, even if agent 2 is rank-envious for the desired agent 3, agent 3 would not prefer to be matched with agent 2.*

The next basic observation can be made for r^{BP}-EF matchings.

Observation 2. *An r^{BP}-EF matching matches every agent ranked first by someone with an agent who ranks her first or with an agent that she ranks better than all agents who rank her first.*

By definition, every BP-stable matching is r^{BP}-EF, implying that an r^{BP}-EF matching always exists under globally-ranked [4] or narcissistically single-peaked preferences [13]. Yet, in general, an r^{BP}-EF matching may not exist, as shown in the next example. Moreover, a wr-EF matching may not exist even under 1-Euclidean preferences.

Example 2. *Let us consider a roommate instance with six agents who have the preferences below.*

$$\begin{array}{lccccccccc}
1: & 4 & \succ & 3 & \succ & 5 & \succ & 2 & \succ & 6 \\
2: & 5 & \succ & 6 & \succ & 1 & \succ & 3 & \succ & 4 \\
3: & 4 & \succ & 2 & \succ & 5 & \succ & 6 & \succ & 1 \\
4: & 1 & \succ & 6 & \succ & 2 & \succ & 5 & \succ & 3 \\
5: & 6 & \succ & 2 & \succ & 3 & \succ & 1 & \succ & 4 \\
6: & 2 & \succ & 5 & \succ & 1 & \succ & 4 & \succ & 3
\end{array}$$

By Observation 2, agent 2 can only be matched with agent 5 or 6, while agent 5 can only be matched with agent 6 or 2 and agent 6 can only be matched with agent 2 or 5. This cannot be achieved together, thus there cannot be an r^{BP}-EF matching in this instance.

Consequently, we further relax wr-EF and r^{BP}-EF by combining their two versions of justified envy. Agent i strongly BP rank-envies agent j in matching σ if $\sigma(j) \succ_i \sigma(i)$, $r_i(\sigma(j)) < r_j(\sigma(j))$, $r_{\sigma(i)}(i) > r_i(\sigma(j))$ and $i \succ_{\sigma(j)} j$.

Definition 5 (Weak BP rank-envy-freeness (wr^{BP}-EF))**.** *A matching σ is wr^{BP}-EF if $\sigma(i) \succ_i \sigma(j)$ or $r_j(\sigma(j)) \leq r_i(\sigma(j))$ or $r_{\sigma(i)}(i) \leq r_i(\sigma(j))$ or $j \succ_{\sigma(j)} i$, for every $i, j \in N$.*

We prove below that a rank-maximal matching is always wr^{BP}-EF, guaranteeing the existence of a wr^{BP}-EF matching.

Proposition 1. *Every rank-maximal matching is wr^{BP}-EF.*

Proof. Let σ be a rank-maximal matching that is not wr^{BP}-EF. It follows that there exist two agents i and j such that $\sigma(j) \succ_i \sigma(i)$, $r_i(\sigma(j)) < r_j(\sigma(j))$, $r_i(\sigma(j)) < r_{\sigma(i)}(i)$ and $i \succ_{\sigma(j)} j$. If we consider the matching σ' resulting from the swap between i and j in σ, then i is matched with $\sigma(j)$ and j is matched with $\sigma(i)$ in σ', while the rest remains unchanged. Therefore, i and $\sigma(j)$ are better off in σ', while $\sigma(i)$ and j may be worse off. However, the rank that i gets for her partner in σ' is strictly better than the one that $\sigma(i)$ and j get in σ. Therefore, the signature of σ' is strictly better, lexicographically, than the signature of σ, contradicting the rank-maximality of σ. ☐

Corollary 1. *Every roommate instance admits a wr^{BP}-EF matching.*

4 Characterization of Rank-Based Envy-Freeness

For a given matching σ and a given integer $k \in [n]$, let N_k^σ denote the subset of agents who are matched in σ with an agent they rank at a position better than k in their preferences, i.e., $N_k^\sigma := \{j \in N : r_j(\sigma(j)) < k\}$. This enables us to define, for each agent $i \in N$, the best rank $\rho^\sigma(i)$ that a "not already satisfied" agent can assign to i, i.e., $\rho^\sigma(i) = \min_{k \in [n-1]} \min_{j \in N \setminus N_k^\sigma} r_j(i)$. Then, we can partition the agents w.r.t. these ranks in subsets $(T_1^\sigma, \ldots, T_{n-1}^\sigma)$ where $T_k^\sigma := \{i \in N : \rho^\sigma(i) = k\}$ for each $k < n$. For each agent i, let A_i^σ be the subset of agents that realize the best rank of i, i.e., $A_i^\sigma = \{j \in N \setminus N_{\rho^\sigma(i)}^\sigma : r_j(i) = \rho^\sigma(i)\}$ (the mentions to σ may be omitted).

We refine Observation 1 in the next characterization theorem.

Theorem 1. *A matching σ is r-EF iff every agent $i \in T_\ell^\sigma$ is assigned to an agent in A_i^σ, for all $\ell \in [n-1]$.*

Proof. Suppose that a matching σ is r-EF. We proceed by induction over $\ell \in [n-1]$. The base case $\ell = 1$ trivially follows from Observation 1. Suppose that the statement is true for all $\ell' < \ell$ for a given $\ell \in [n-1]$, and that there exists an agent $i \in T_\ell$ such that $\sigma(i) \notin A_i$. By definition of T_ℓ, $A_i \neq \emptyset$, and we can consider an agent $j \in A_i$. Since $\sigma(i) \notin A_i$, we have $\sigma(i) \in N_\ell^\sigma$ or $r_{\sigma(i)}(i) \neq \ell$. If $\sigma(i) \in N_\ell^\sigma$, then by definition $r_{\sigma(i)}(i) < \ell$ implying that $i \in T_{\ell'}$ for some $\ell' < \ell$, a contradiction. Otherwise, we must have $r_{\sigma(i)}(i) > \ell$, implying that j rank-envies $\sigma(i)$ since $r_j(i) = \ell$ and $j \notin N_\ell^\sigma$, a contradiction.

Consider now the matching σ that assigns to each agent $i \in T_\ell$ an agent in A_i, for every $\ell \in [n-1]$. Suppose that σ is not r-EF. There exist agents i and j such that i rank-envies j, i.e., $r_i(\sigma(j)) < r_i(\sigma(i))$ and $r_i(\sigma(j)) < r_j(\sigma(j))$. Suppose that $\sigma(j) \in T_\ell$, i.e., $r_j(\sigma(j)) = \ell$, and thus $\ell' := r_i(\sigma(j)) < \ell$. Since $\sigma(j) \notin T_{\ell'}$, we have that $i \in N^\sigma_{\ell'}$ and thus $r_i(\sigma(i)) < \ell'$, a contradiction. ☐

Let us now generalize this characterization by considering rank-envy-freeness parameterized by rank k. For a given matching σ, let us denote by E_i the subset of agents that prefer i to their match or are matched with i, i.e., $E_i := \{j \in N : \exists \ell, r_j(i) = \ell \wedge j \notin N^\sigma_\ell\}$.

Proposition 2. *A matching σ is r_k-EF iff for every agent $i \in T_\ell$ we have $\sigma(i) \in A_i$, for all $\ell \in [n-1]$, and $|E_i| = 1$ if $\ell > k$.*

The previous characterization for r_k-EF becomes more interesting when $k = 1$ since r_1-EF gets the same characterization as popularity in the house allocation setting [3].

Proposition 3. *A matching is r_1-EF iff every agent gets matched with either her most preferred agent or her most preferred agent that is not ranked first by someone.*

Proof. Let σ be a matching under which one of the two conditions of the statement holds. Suppose that there exist agents i and j such that i r_1-envies j, i.e., $\sigma(j) \succ_i \sigma(i)$ and $[r_j(\sigma(j)) > 1$ or $r_j(\sigma(j)) > r_i(\sigma(i))]$ (*). Since i is envious, $\sigma(i)$ cannot be her most preferred agent, and thus $\sigma(i)$ is her most preferred agent that is not ranked first by someone. It follows that $\sigma(j)$ is ranked first by someone. It cannot be by agent j because, otherwise, condition (*) would not hold. Therefore, $\sigma(j)$ is ranked first by someone else, a contradiction.

Let σ be an r_1-EF matching. Consider any agent i. If $\sigma(i)$ is the most preferred agent of i, we are done. Otherwise, i envies all agents j such that $\sigma(j) \succ_i \sigma(i)$. Since it cannot be r_1-envy, $r_j(\sigma(j)) \leq \min\{r_i(\sigma(j)), 1\} = 1$ holds for all such j. Suppose now that $\sigma(i)$ is also ranked first by some agent j. Therefore, $\sigma(j)$ is not the most preferred agent of j and thus j r_1-envies i, a contradiction. ☐

Considering relaxations of rank-envy-freeness, we can provide some necessary conditions for r^{BP}-EF and wr-EF, stated below. Finding exact characterizations for these criteria is an open question.

Proposition 4. *If a matching σ is r^{BP}-EF then for every $i \in T_\ell$ either $\sigma(i) \in A_i$ or $r_i(\sigma(i)) < \min_{j \in A_i} r_i(j)$ holds, for all $\ell \in [n-1]$.*

Observation 3. *For every i in T_1 such that for every $j \in A_i$, $j \notin T_1$ or $A_j = \{i\}$, we have $\sigma(i) \in A_i$, in every wr-EF matching.*

5 Existence of Rank-Based Envy-Free Matchings

We have already mentioned some existence statements when introducing our criteria. This section deals with the complexity of deciding the existence of a solution satisfying a given fairness concept.

An r-EF matching does not always exist, as shown in Example 1, but we further prove below that the related decision problem is hard.

Theorem 2. *Deciding whether a roommate instance admits an r-EF matching is NP-complete even under globally-ranked preferences.*

Proof sketch. Membership to NP is straightforward. For hardness, we reduce from (3,B2)-SAT [10]. In an instance of (3,B2)-SAT, we are given a CNF propositional formula ψ where every clause C_j, for $j \in [m]$, contains exactly three literals and every variable x_i, for $i \in [p]$, occurs exactly twice as a positive literal x_i and twice as a negative literal $\overline{x}_i$ in ψ. From an instance of (3,B2)-SAT, we construct a roommate instance as follows.

- For each clause C_j, we create: three clause-agents K^1_j, K^2_j, and K^3_j, associated with the three literals of the clause, two agents q^1_j and q^2_j, and two agents G^1_j and G^2_j.
- For each variable x_i, we create: four literal-agents y^1_i, y^2_i, $\overline{y}^1_i$ and $\overline{y}^2_i$ and four literal-agents Z^1_i, Z^2_i, $\overline{Z}^1_i$ and $\overline{Z}^2_i$, associated with each occurrence of variable literals, two agents a_i and b_i, and a variable-gadget composed of six agents Γ_i, Δ_i, Θ_i and γ_i, δ_i, θ_i.
- We additionally create $\frac{m}{2}$ agents d_k for $k \in [\frac{m}{2}]$, and ten dummy agents λ_k and Λ_k for $k \in [5]$.

The preferences of the agents are constructed as described in Table 1.[1] One can observe that the preferences are globally-ranked.

Table 1: Agents' preferences for the reduction proof of Theorem 2 for every $j \in [m]$, $i \in [p]$, $k \in [m/2]$, and $r \in [5]$. Notation t^ℓ_j stands for the literal-agent associated with the ℓ^{th} literal of clause C_j for $\ell \in [3]$; and $K(y^l_i)$ (resp., $K(\overline{y}^l_i)$) denotes the clause-agent K^ℓ_j such that $t^\ell_j = y^l_i$ (resp., $t^\ell_j = \overline{y}^l_i$). Notation $[x]_{condition}$ means that x is present at the given rank only if *condition* is satisfied. Framed dots $[\dots]$ denote an arbitrary order over the remaining agents.

$$K^1_j: \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ t^1_j \succ q^1_j \succ \lambda_4 \succ [\dots]$$
$$K^2_j: \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ t^2_j \succ \lambda_4 \succ q^2_j \succ [\dots]$$
$$K^3_j: \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ t^3_j \succ q^1_j \succ q^2_j \succ [\dots]$$

$$G^1_j: \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4 \succ q^1_j \succ \lambda_5 \succ d_1 \succ \dots \succ d_{\frac{m}{2}} \succ [\dots]$$
$$G^2_j: \lambda_1 \succ \lambda_2 \succ \lambda_3 \succ \lambda_4 \succ \lambda_5 \succ q^2_j \succ d_1 \succ \dots \succ d_{\frac{m}{2}} \succ [\dots]$$

$$Z^1_i: a_i \succ \delta_i \succ y^1_i \succ [\dots]$$
$$\overline{Z}^1_i: a_i \succ \theta_i \succ \overline{y}^1_i \succ [\dots]$$
$$Z^2_i: b_i \succ \delta_i \succ y^2_i \succ [\dots]$$
$$\overline{Z}^2_i: b_i \succ \theta_i \succ \overline{y}^2_i \succ [\dots]$$

$$\Gamma_i: \lambda_1 \succ \lambda_2 \succ \theta_i \succ \delta_i \succ [\dots]$$
$$\Delta_i: \gamma_i \succ \delta_i \succ \theta_i \succ [\dots]$$
$$\Theta_i: \gamma_i \succ \theta_i \succ \delta_i \succ [\dots]$$

$$\Lambda_r: \lambda_r \succ [\dots]$$

$$q^1_j: K^1_j \succ K^3_j \succ G^1_j \succ [\dots]$$
$$q^2_j: K^2_j \succ K^3_j \succ G^2_j \succ [\dots]$$

$$d_k: \Lambda_2 \succ \Lambda_1 \succ G^1_1 \succ G^2_1 \succ \dots \succ G^1_m \succ G^2_m \succ [\dots]$$

$$y^1_i: [\Lambda_2]_{\exists j,\, y^1_i = t^3_j} \succ K(y^1_i) \succ [\Lambda_2]_{\exists j, y^1_i \in \{t^1_j, t^2_j\}} \succ Z^1_i \succ [\dots]$$
$$\overline{y}^1_i: [\Lambda_2]_{\exists j,\, \overline{y}^1_i = t^3_j} \succ K(\overline{y}^1_i) \succ [\Lambda_2]_{\exists j, \overline{y}^1_i \in \{t^1_j, t^2_j\}} \succ \overline{Z}^1_i \succ [\dots]$$
$$y^2_i: [\Lambda_2]_{\exists j,\, y^2_i = t^3_j} \succ K(y^2_i) \succ [\Lambda_2]_{\exists j, y^2_i \in \{t^1_j, t^2_j\}} \succ Z^2_i \succ [\dots]$$
$$\overline{y}^2_i: [\Lambda_2]_{\exists j,\, \overline{y}^2_i = t^3_j} \succ K(\overline{y}^2_i) \succ [\Lambda_2]_{\exists j, \overline{y}^2_i \in \{t^1_j, t^2_j\}} \succ \overline{Z}^2_i \succ [\dots]$$

$$a_i: \Lambda_2 \succ \Lambda_3 \succ \Lambda_4 \succ Z^1_i \succ \overline{Z}^1_i \succ [\dots]$$
$$b_i: \Lambda_2 \succ \Lambda_3 \succ \Lambda_4 \succ Z^2_i \succ \overline{Z}^2_i \succ [\dots]$$

$$\gamma_i: \Gamma_i \succ \Delta_i \succ \Theta_i \succ [\dots]$$
$$\delta_i: \Gamma_i \succ \Delta_i \succ \Theta_i \succ [\dots]$$
$$\theta_i: \Gamma_i \succ \Theta_i \succ \Delta_i \succ [\dots]$$

$$\lambda_r: \Lambda_r \succ [\dots]$$

We claim that formula ψ is satisfiable iff there exists an r-EF matching in the constructed roommate instance.

$\Longrightarrow$: Suppose that there exists a truth assignment ϕ of the variables that satisfies ψ. We construct an r-EF matching σ as follows:

- For every $r \in [5]$, let $\sigma(\lambda_r) = \Lambda_r$;

[1] One can show that this reduction proof also holds for the marriage setting, by distinguishing the lower and upper case agents as the two different types.

- For every literal-agent y_i^ℓ (resp., $\overline{y}_i^\ell$), if the associated literal x_i (resp., $\overline{x}_i$) is true in ϕ then $\sigma(y_i^\ell) = K(y_i^\ell)$ (resp., $\sigma(\overline{y}_i^\ell) = K(\overline{y}_i^\ell)$). Otherwise, $\sigma(y_i^\ell) = Z_i^\ell$ (resp., $\sigma(\overline{y}_i^\ell) = \overline{Z}_i^\ell$);
- For every clause-agent K_j^ℓ that has not been assigned yet, if $\ell \in [2]$, then $\sigma(K_j^\ell) = q_j^\ell$, otherwise $\sigma(K_j^3) = q_j^1$ if q_j^1 is still available or $\sigma(K_j^3) = q_j^2$ otherwise;
- For every agent Z_i^ℓ (resp., $\overline{Z}_i^\ell$) that has not been assigned yet, $\sigma(Z_i^\ell) = a_i$ if $\ell = 1$ and $\sigma(Z_i^\ell) = b_i$ if $\ell = 2$ (resp., $\sigma(\overline{Z}_i^\ell) = a_i$ if $\ell = 1$ and $\sigma(\overline{Z}_i^\ell) = b_i$ if $\ell = 2$).
- For every variable x_i, if x_i is true in ϕ then let $\sigma(\Delta_i) = \gamma_i$, $\sigma(\Theta_i) = \theta_i$, and $\sigma(\Gamma_i) = \delta_i$. Otherwise, i.e., if x_i is false in ϕ, then let $\sigma(\Delta_i) = \delta_i$, $\sigma(\Theta_i) = \gamma_i$, and $\sigma(\Gamma_i) = \theta_i$.
- For every agent q_j^ℓ that has not been assigned yet, let $\sigma(q_j^\ell) = G_j^\ell$.
- For every remaining agents G_j^ℓ and d_k, let $\sigma(G_j^\ell) = d_k$.

$\Longleftarrow$: The global idea is as follows. For each $i \in [p]$, the variable-gadget enforces that exactly two literal-agents among $y_i^1, y_i^2, \overline{y}_i^1$, and $\overline{y}_i^2$ are matched with their associated agents Z_i^1, Z_i^2, $\overline{Z}_i^1$, and $\overline{Z}_i^2$, and that these two chosen literal-agents mimic a truth assignment of the variables, i.e., they correspond to the same literal. The two remaining literal-agents are forced to be matched with their associated clause-agent, representing the clause to which their corresponding occurrence of literal belongs. If for a clause C_j, no clause-agent K_j^ℓ is assigned to a literal-agent, then agent K_j^3 will be necessarily assigned to an agent ranking her worse than second, while her associated literal-agent t_j^3 ranks her second and is currently matched with her third choice, creating rank-envy, a contradiction. □

Whereas the characterization results of the previous section enable to derive algorithms that can be exponential, the characterization of r_1-EF provides a polynomial-time algorithm, as stated below.

Proposition 5. *Deciding whether a roommate instance admits an r_1-EF matching can be done in polynomial time.*

Proof. By Proposition 3, it suffices to search for a perfect matching in the undirected graph $G = (N, E)$ where $\{i, j\} \in E$ iff j is either ranked first by i or is her most preferred agent that is not ranked first by someone, and the reverse holds for i w.r.t. j's preferences. □

Concerning the relaxations of rank-envy-freeness, the same construction as in the proof of Theorem 2 can be used to show that deciding about the existence of a wr-EF matching is NP-hard.

Proposition 6. *Deciding whether a roommate instance admits a wr-EF matching is NP-complete under globally-ranked preferences.*

Although an r^{BP}-EF matching can be efficiently computed by constructing a BP-stable matching, under globally-ranked or narcissistically single-peaked preferences [4, 13], we prove that in general the related decision problem is hard. The complexity proof is similar to the one in Theorem 2 but the agents cannot be partitioned in two types and we need gadgets that are based on odd rings [15].

Theorem 3. *Deciding whether a roommate instance admits an r^{BP}-EF matching is NP-complete.*

Finally, by the fact that a rank-maximal matching always exists in a roommate instance, the same guarantee holds for a wr^{BP}-EF matching (Corollary 1). As already discussed in the literature, a rank-maximal matching can be constructed by solving a maximum weight matching problem with weight $n^{n-r_i(j)} + n^{n-r_j(i)}$ for a pair of agents $\{i, j\}$. By using the scaling algorithm of Gabow and Tarjan [21], this problem can be solved in polynomial time.

Corollary 2. *A wr^{BP}-EF matching can be found in polynomial time.*

6 Rank-Based Envy-Freeness and Optimality

Let us analyze in this section the connections between optimal matchings and rank-based envy-free matchings, that result in the links between optimality and fairness criteria of Figure 1.

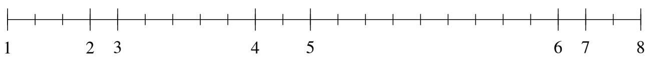
We first prove that, although rank-maximality implies our weakest criterion wr^{BP}-EF (Proposition 1), it is not true for popularity.

Proposition 7. *A popular matching may not be wr^{BP}-EF, even under 1-Euclidean preferences.*

Proof. Let us consider a roommate instance with eight agents who have the preferences below.

1 :	2	≻	3	≻	[4]	≻	5	≻	6	≻	7	≻	8
2 :	[3]	≻	1	≻	4	≻	5	≻	6	≻	7	≻	8
3 :	[2]	≻	1	≻	4	≻	5	≻	6	≻	7	≻	8
4 :	5	≻	3	≻	2	≻	[1]	≻	6	≻	7	≻	8
5 :	4	≻	3	≻	2	≻	6	≻	7	≻	1	≻	[8]
6 :	[7]	≻	8	≻	5	≻	4	≻	3	≻	2	≻	1
7 :	[6]	≻	8	≻	5	≻	4	≻	3	≻	2	≻	1
8 :	7	≻	6	≻	[5]	≻	4	≻	3	≻	2	≻	1

The preferences are 1-Euclidean w.r.t. the following embedding:



The framed matching is popular but not wr^{BP}-EF: agent 4 is strongly BP rank-envious towards agent 8. □

Note that the implication of wr^{BP}-EF by rank-maximality is tight w.r.t. the hierarchy of our relaxed criteria because stronger fairness criteria are not implied by rank-maximality, as stated below.

Proposition 8. *A rank-maximal matching may not be wr-EF or r^{BP}-EF, even under 1-Euclidean preferences.*

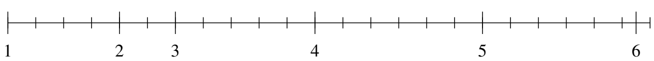
Conversely, even r_1-EF is not a stronger requirement than rank-maximality, as proved in the next proposition.

Proposition 9. *An r_1-EF matching may not be rank-maximal even under 1-Euclidean preferences.*

Proof. Let us consider a roommate instance with six agents who have the preferences below.

1 :	[2]	≻	3	≻	(4)	≻	5	≻	6
2 :	(3)	≻	[1]	≻	4	≻	5	≻	6
3 :	(2)	≻	[4]	≻	1	≻	5	≻	6
4 :	[3]	≻	5	≻	2	≻	(1)	≻	6
5 :	[(6)]	≻	4	≻	3	≻	2	≻	1
6 :	[(5)]	≻	4	≻	3	≻	2	≻	1

The preferences are 1-Euclidean w.r.t. the following embedding:



The encircled matching is r_1-EF, but it is not rank-maximal because the unique rank-maximal matching is the framed matching. □

And even r_2-EF is not stronger than popularity, as stated below.

Proposition 10. *An r_2-EF matching may not be popular even under 1-Euclidean preferences.*

The previous result is tight w.r.t. the scale of r_k-EF because it does not hold when $k = 1$, as proved in the next proposition.

Proposition 11. *Every r_1-EF matching is popular.*

Proof. Let σ^1 be an r_1-EF matching that is not popular, i.e., there exists a matching σ^2 such that $|N^2| > |N^1|$, where $N^\ell := \{i \in N : \sigma^\ell(i) \succ_i \sigma^{3-\ell}(i)\}$ for $\ell \in [2]$. Let i be an agent in N^2, and j the agent such that $\sigma^2(i) = \sigma^1(j)$. Since σ^1 is r_1-EF, we must have that $r_j(\sigma^1(j)) \leq \min\{r_i(\sigma^1(j)), 1\} = 1$. Therefore, $\sigma^1(j) \succ_j \sigma^2(j)$, and thus $j \in N^1$. Hence, we can associate each agent in N^2 with a distinct agent in N^1, and thus $|N^1| \geq |N^2|$, a contradiction. □

Finally, our weakest fairness criterion satisfies Pareto-optimality.

Proposition 12. *Every wr^{BP}-EF matching is Pareto-optimal.*

Proof. Suppose that a wr^{BP}-EF matching σ is Pareto-dominated by matching σ'. Therefore, there exists an improving cycle from σ to σ' along the agents $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ such that $\sigma'(i_\ell) = \sigma(i_{\ell+1}) = j_{\ell+1}$ and $\sigma'(j_\ell) = \sigma(j_{\ell-1}) = i_{\ell-1}$ for every $\ell \in [k]$ ($k+1$ corresponds to 1 and 0 corresponds to k). Consider an agent $c_1 := i_1$ in the improving cycle. Since σ' Pareto-dominates σ, we have $\sigma'(i_\ell) = j_{\ell+1} \succ_{i_\ell} \sigma(i_\ell) = j_\ell$ and $\sigma'(j_\ell) = i_{\ell-1} \succ_{j_\ell} \sigma(j_\ell) = i_\ell$ for every $\ell \in [k]$. Therefore, the agents i_1 and j_2, who are matched together in σ', form a blocking pair in σ and agent i_1 envies agent i_2 who is matched with agent j_2 in σ. Thus, for i_1 not being strongly BP rank-envious towards i_2, we need that (1) $r_{i_2}(j_2) \leq r_{i_1}(j_2)$ or (2) $r_{j_1}(i_1) \leq r_{i_1}(j_2)$. If (1) holds, then we have that $r_{i_2}(\sigma'(i_2)) < r_{i_2}(\sigma(i_2)) = r_{i_2}(j_2) \leq r_{i_1}(j_2) = r_{i_1}(\sigma'(i_1))$. If (2) holds, then we have that $r_{j_1}(\sigma'(j_1)) < r_{j_1}(\sigma(j_1)) = r_{j_1}(i_1) \leq r_{i_1}(j_2) = r_{i_1}(\sigma'(i_1))$. If (1) holds then we set $c_2 := i_2$, otherwise (2) holds and we set $c_2 := j_1$. It follows that c_2 is matched with a partner at a rank strictly better than c_1 in σ'. By repeating this argument starting from c_2 and so on, we get a chain of agents $c_1, c_2, \ldots, c_{k'}$ from $\bigcup_{\ell \in [k]} \{i_\ell, j_\ell\}$ such that c_ℓ is matched with a partner at a rank strictly better than $c_{\ell-1}$ in σ', for $\ell \in \{2, \ldots, k\}$. Since $\bigcup_{\ell \in [k]} \{i_\ell, j_\ell\}$ is finite, there exist two agents c_ℓ and $c_{\ell'}$ for $\ell \neq \ell'$ such that $c_\ell = c_{\ell'}$, contradicting the iterative strict improvement of the ranks in σ'. □

7 Rank-Based Envy-Freeness and Stability

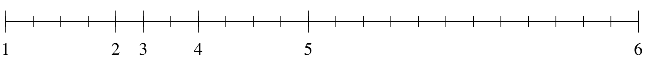
We analyze in this section the connections between stable matchings and rank-based envy-free matchings, that result in the links between stability and fairness criteria of Figure 1. We first remark that, although BP-stability implies r^{BP}-EF and thus the criterion wr^{BP}-EF, it does not imply further rank-based envy-free criteria.

Proposition 13. *A blocking-pair and swap-stable matching may not be wr-EF even under 1-Euclidean preferences.*

Proof. Let us consider a roommate instance with six agents who have the preferences below.

1 :	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	$\boxed{6}$
2 :	$\boxed{3}$	$\succ$	4	$\succ$	1	$\succ$	5	$\succ$	6
3 :	$\boxed{2}$	$\succ$	4	$\succ$	1	$\succ$	5	$\succ$	6
4 :	3	$\succ$	2	$\succ$	$\boxed{5}$	$\succ$	1	$\succ$	6
5 :	$\boxed{4}$	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	6
6 :	5	$\succ$	4	$\succ$	3	$\succ$	2	$\succ$	$\boxed{1}$

The preferences are 1-Euclidean w.r.t. the following embedding:



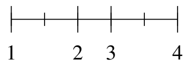
The framed matching is BP-stable and swap-stable but it is not wr-EF: agent 6 envies agent 4 for her partner 5. □

Conversely, even r_1-EF does not imply BP-stability.

Proposition 14. *An r_1-EF matching may not be blocking-pair stable even under 1-Euclidean preferences.*

Proof. Let us consider a roommate instance with four agents who have the preferences below (left) that are 1-Euclidean w.r.t. the embedding given below (right).

1 :	$\boxed{2}$	$\succ$	3	$\succ$	4
2 :	3	$\succ$	$\boxed{1}$	$\succ$	4
3 :	2	$\succ$	$\boxed{4}$	$\succ$	1
4 :	$\boxed{3}$	$\succ$	2	$\succ$	1



The framed matching is r_1-EF but it is not BP-stable: agents 2 and 3 prefer to be together than with their current partner. □

Every envy-free matching is trivially swap-stable because no agent prefers the partner of another agent. We show that although r-EF is a strong relaxation of envy-freeness, it still implies swap-stability.

Proposition 15. *Every r-EF matching is swap-stable.*

Proof. Suppose that two agents i and j have an incentive to swap from a matching σ which is r-EF, i.e., $r_i(\sigma(j)) < r_i(\sigma(i))$ and $r_j(\sigma(i)) < r_j(\sigma(j))$. By rank-envy-freeness, we have $r_j(\sigma(j)) \leq r_i(\sigma(j))$. Therefore, we get that $r_j(\sigma(i)) < r_j(\sigma(j)) \leq r_i(\sigma(j)) < r_i(\sigma(i))$, and thus $r_j(\sigma(i)) < r_i(\sigma(i))$, implying that agent j is rank-envious towards agent i in σ, a contradiction. □

However, this implication does not hold for weaker versions of rank-envy-freeness, as stated below.

Proposition 16. *A wr-EF matching may not be swap-stable even under 1-Euclidean preferences.*

Proposition 17. *An r^{BP}-EF matching may not be swap-stable even under globally-ranked or narcissistically single-peaked preferences.*

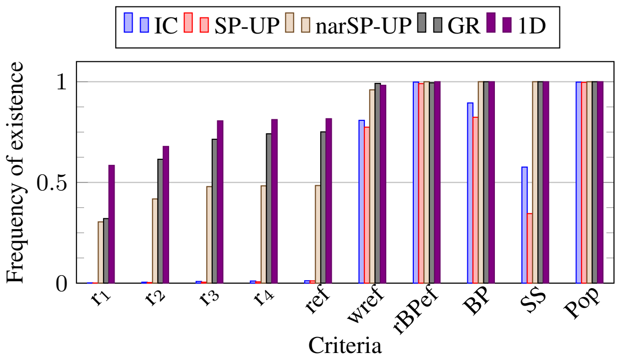
Yet, under stronger restrictions, the implication holds for r^{BP}-EF.

Proposition 18. *Every r^{BP}-EF matching is swap-stable under 1-Euclidean preferences.*

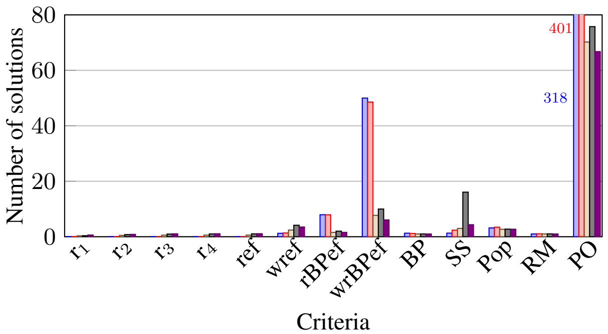
Proof. Let σ be an r^{BP}-EF matching that is not swap-stable. There exist two agents i and j such that $\sigma(j) \succ_i \sigma(i)$ and $\sigma(i) \succ_j \sigma(j)$. There must exist an agent between i and j who rank the partner of the other agent at a better rank, say it is the case for j, i.e., $r_j(\sigma(i)) < r_i(\sigma(i))$. Because σ is r^{BP}-EF, we must have $i \succ_{\sigma(i)} j$. Suppose that the preferences are 1-Euclidean w.r.t. the embedding $E : N \to \mathbb{R}$ such that, w.l.o.g., $E(i) < E(j)$. To respect the preferences of the agents, only two orders are possible: (1) $E(i) < E(\sigma(j)) < E(\sigma(i)) < E(j)$, or (2) $E(\sigma(j)) < E(i) < E(\sigma(i)) < E(j)$.

If (1) holds, then $E(\sigma(i)) - E(i) < E(j) - E(\sigma(i))$ because $i \succ_{\sigma(i)} j$. Since $E(\sigma(j)) < E(\sigma(i))$, we have that $E(\sigma(j)) - E(i) < E(j) - E(\sigma(j))$, i.e., agent $\sigma(j)$ prefers i to j. By the axis, $\sigma(j)$ also prefers $\sigma(i)$ to j. If $\sigma(j)$ prefers $\sigma(i)$ to i (resp., i to $\sigma(i)$), then the only agents that $\sigma(j)$ may prefer to $\sigma(i)$ (resp., i) are the agents $x \neq \sigma(j)$ such that $E(i) < E(x) < E(\sigma(i))$. In contrast, i (resp., $\sigma(i)$) prefers the agents x to $\sigma(i)$ (resp., i) such that $E(i) < E(x) < E(\sigma(i))$, including $\sigma(j)$. Therefore, $\sigma(j)$ rank-envies i (resp., $\sigma(i)$), while agent $\sigma(i)$ (resp., i) also prefers $\sigma(j)$ to i (resp., $\sigma(i)$), i.e., $\sigma(i)$ (resp., i) and $\sigma(j)$ form a blocking pair, a contradiction.

If (2) holds, then we have that the only agents that i may prefer to $\sigma(j)$ are the agents $x \neq i$ such that $E(\sigma(j)) < E(x) < E(\sigma(i))$. In contrast, j prefers the agents x to $\sigma(j)$ such that $E(\sigma(j)) < E(x) \leq E(\sigma(i))$ or $E(\sigma(i)) < E(x) < E(j)$. Therefore, i is rank-envious towards j, while agent $\sigma(j)$ also prefers i over j (by the axis), meaning that i and $\sigma(j)$ form a blocking pair, a contradiction. □



(a) Proportion of instances for which a matching satisfying the given criterion exists (criteria for which existence is guaranteed are omitted)



(b) Number of matchings in average that satisfy the given criterion

Figure 2: Existence experiments for rank-based envy-free matchings, stable matchings, or optimal matchings in roommate instances with 10 agents and various preference distributions for 10,000 runs.

8 Empirical Study of Rank-Based Envy-Freeness

In this section, we empirically evaluate the existence of fair matchings, and compare them to optimal or stable matchings, under different types of preferences. The generated preferences can follow: the *impartial culture (IC)*, i.e., they are uniformly drawn from the set of all possible linear orders; the *(narcissistically) single-peaked uniform peak ((nar)SP-UP)* distribution [16], i.e., an axis $>$ over agents is generated and then, for each agent i, we construct $\succ_i$ by uniformly choosing a peak agent and then iteratively and uniformly choosing between the left and right next available agent on $>$ for the next preferred agent (for narcissistic preferences, the peak is one of the two agents directly adjacent to i in $>$); the *globally-ranked uniform (GR)* distribution, i.e., the preference profile is derived from a uniformly drawn global order $\rhd$ over all pairs of agents; or the *1-Euclidean (1D)* uniform distribution, i.e., a preference profile is generated by first uniformly choosing a real number for each agent and then deriving all agents' preference rankings by sorting the distances.

The existence of matchings satisfying different criteria is analyzed, for $n = 10$ agents and by sampling $10,000$ instances for each preference distribution. The criteria under consideration are rank-based envy-freeness, where the results for r_k-EF are only shown for $k \in [4]$, BP-stability (BP), swap-stability (SS), popularity (Pop), rank-maximality (RM), and Pareto-optimality (PO). For each criterion, we count the number of instances where a matching satisfying it exists as well as the number of such matchings per instance and compute its average (all possible matchings are checked).

Figure 2a shows the proportion of instances that satisfy each criterion. We observe that r-EF and r_k-EF matchings exist with a significant probability only for preferences that are at least narcissistically single-peaked or globally-ranked, they rarely exist for general or single-peaked preferences. This confirms that r-EF is a very demanding fairness criterion. However, for 1-Euclidean preferences, even an r_1-EF matching exists with a probability greater than 0.5, so we really need a strong structure on the preferences. Interestingly, a wr-EF matching exists with a probability greater than 0.75 for all preference distributions (and even greater than 0.9 for narSP-UP or GR distributions). This finding contrasts with our theoretical results establishing the same impossibilities for r-EF and wr-EF, and shows that wr-EF is nevertheless a useful relaxation. Finally, the probability of existence of an r^{BP}-EF matching is very close to 1 even on general preferences for which the existence is not theoretically guaranteed.

Figure 2b indicates the average number of matchings that satisfy each criterion. As expected, the number of r-EF matchings is very low. While there are more wr-EF matchings, this number is not very large, even for instances where there exist such matchings, showing that this concept would not recommend too many solutions in practice. The same remark holds for r^{BP}-EF. A larger number of matchings satisfy wr^{BP}-EF, especially under weak preference restrictions. Nevertheless, this number is much smaller than the number of Pareto-optimal matchings, showing the filtering ability of wr^{BP}-EF.

9 Conclusion

We have investigated fair roommate matchings under the prism of rank-based envy-freeness. Our study takes inspiration from a previous work on rank-envy-freeness (r-EF) in house allocation [9]. Interestingly, when matching agents to other agents instead of matching them to objects, we observe important differences w.r.t. fairness. Indeed, while an r-EF matching can be computed in polynomial time in house allocation, we have shown that r-EF may not be achievable in the roommate setting and that the related decision problem is NP-complete. Additionally, whereas the stronger criterion r_1-EF is equivalent to popularity in house allocation [9], we show that their characterization in that setting [3] fails to transfer to the roommate setting for popularity but still holds true for r_1-EF. We also connect r-EF and Pareto-optimality, contrary to house allocation, because even our strongest relaxation of r-EF, namely wr^{BP}-EF, implies it, while a wr^{BP}-EF matching is computable in polynomial time. This criterion is nevertheless not trivially satisfied since it discards many Pareto-optimal matchings in practice. See Figure 1 for a global picture of the interactions between stability, optimality and fairness criteria.

Several challenging open questions remain, such as the exact characterization of our relaxations of r-EF. Moreover, whereas deciding about the existence of a weak r-EF (wr-EF) or an r-EF matching is hard even under globally-ranked preferences, we do not know whether other restrictions would help for an efficient algorithm. Finally, although the existence of an r_k-EF matching can be verified in polynomial time when $k = 1$, we conjecture that it is hard for $k = 2$.

Our study raises many interesting perspectives for future work. It would for instance make sense to explore settings with indifference or partial preference lists. However, since our results already point out many impossibilities, there is no hope to get much more positive results in such extensions. For instance, even in the marriage setting, a wr-EF matching may not exist under 1-Euclidean preferences, and the complexity proof of Theorem 2 also holds. The main difference is that an r^{BP}-EF matching can always be computed in polynomial time, because it is the case for a BP-stable matching [22]. In contrast, the related existence problem is NP-complete in the roommate setting.

Acknowledgements

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